

Research Laboratory
of the
Portland Cement Association

BULLETIN 6

Dynamic Testing of Pavements

BY
GERALD PICKETT

APRIL, 1945

CHICAGO

Authorized Reprint from Copyrighted
JOURNAL OF THE AMERICAN CONCRETE INSTITUTE
New Center Building, 7400 Second Boulevard
Detroit 2, Michigan
April 1945, PROCEEDINGS V. 41, p. 473

JOURNAL
of the
AMERICAN CONCRETE INSTITUTE
(copyrighted)

Vol. 16 No. 5

7400 SECOND BOULEVARD, DETROIT 2, MICHIGAN

April 1945

Dynamic Testing of Pavements*

By GERALD PICKETT†

SYNOPSIS

A theoretical analysis is made of the problem of the vibration of a pavement in contact with an elastic subgrade. The analysis shows that for each frequency of sustained vibration, waves may be propagated horizontally with three different velocities. The properties of the subgrade have very little effect on the highest or the lowest of these three velocities but have considerable effect on the intermediate velocity. Near the origin the velocity of each wave may be greater or less than the velocity farther from the origin. The analysis indicates that the properties of both pavement and subgrade may be determined for any small region of the pavement by placing the source of sustained vibrations in that region.

INTRODUCTION

The dynamic method of determining Young's modulus of elasticity of concrete has been used for several years as a laboratory test.⁽¹⁻⁴⁾ A specimen under test is caused to vibrate at one or more of its resonant frequencies while being supported either on rubber or in such a way that the support has a negligible effect upon the results. Equations giving the approximate value of these resonant frequencies in terms of the weight, elastic constants, and dimensions of the specimen are available for the usual shapes of laboratory specimens, and thus the modulus of elasticity can be readily computed after the resonant frequency has been determined experimentally.

The application of dynamic testing to concrete pavements and structures in place has probably been retarded both by lack of suitable measuring instruments and by the lack of the necessary mathematical expressions for interpreting the results of measurements. Bernhard⁽⁵⁾ compared the vibration properties of pavement slabs of different thicknesses. He used a mechanical oscillator with a frequency range of from $\frac{1}{2}$ to

*Received by the Institute Dec. 20, 1944.

†Portland Cement Assn., Research Laboratory.

(1-4) See references at end of paper.

40 cycles per second. Recently, Long, Kurtz, and Sandenaw^(6,7) have developed instruments that give reasonable accuracy for the time of travel of waves between two points on the surface of a pavement slab, even when the points are no more than a foot apart. They measured velocity of waves produced by impact and also those produced by sustained vibration at frequencies ranging from 1000 to 2000 cycles per second. They also made some progress in interpreting the results in terms of Young's modulus of elasticity of the pavements by the use of equations developed by Lamb⁽⁸⁾ and Timoshenko.⁽⁹⁾

The accomplishments noted above and the fact that the use of the equations of Lamb and Timoshenko for this purpose may be questionable stimulated the author to prepare the present paper.

LIMITATIONS OF PREVIOUS EQUATIONS

The equations of Lamb and Timoshenko are not strictly applicable to pavements because:

(1) They are for freely vibrating slabs, whereas the pavement is in contact with the subgrade.

(2) They are for a two-dimensional problem (plane waves), whereas the vibration of a pavement is a three-dimensional problem.

In general both the shape and the support of field structures are such as to make analysis of vibration more complicated than that for small laboratory specimens.

So far as is known, the nearest analytical approach to the vibration of a pavement was a study made by Love⁽¹⁰⁾ in which he investigated the vibration of the earth's crust, assuming that the elastic properties of the crust differ from those of the interior. However, Love's studies were also confined to two-dimensional problems and only dealt with cases in which the crust was less rigid than the interior, whereas a concrete pavement will usually be more rigid than its subgrade.

SCOPE OF PRESENT DISCUSSION

This paper will discuss certain possible modes of vibration of a pavement in contact with its subgrade that are likely to occur in the dynamic testing of pavements. An equation giving the relation between driving frequency, thickness of slab, elastic constants of both pavement and subgrade, and the velocity of wave is derived on the assumption of continuity of motion between pavement and subgrade. The equations of elasticity applicable to a homogeneous, isotropic, elastic solid are used for both the slab and the subgrade.

If experiments prove that the equations are generally applicable to pavements in place, then from a few dynamic measurements it should

be possible to determine not only Young's modulus for the concrete but also the thickness of the pavement and a modulus for the subgrade.

PARTICULAR SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF VIBRATION

When expressed in the cylindrical coordinates r, θ, z the differential equations of vibration for a homogeneous, isotropic, elastic solid are:⁽¹¹⁾

$$(\lambda + 2G) \frac{\partial \Delta}{\partial r} - \frac{2G}{r} \frac{\partial W}{\partial \theta} + 2G \frac{\partial V}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$(\lambda + 2G) \frac{1}{r} \frac{\partial \Delta}{\partial \theta} - 2G \frac{\partial U}{\partial z} + 2G \frac{\partial W}{\partial r} = \rho \frac{\partial^2 v}{\partial t^2}$$

$$(\lambda + 2G) \frac{\partial \Delta}{\partial z} - \frac{2G}{r} \frac{\partial}{\partial r} (rV) + \frac{2G}{r} \frac{\partial U}{\partial \theta} = \rho \frac{\partial^2 w}{\partial t^2}$$

where u, v, w are displacements in the r, θ, z directions, respectively; t is time;

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z};$$

$$2U = \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z};$$

$$2V = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r};$$

$$2W = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r} \frac{\partial u}{\partial \theta};$$

$$\rho = \text{mass density};$$

$$\lambda = \frac{\mu E}{(1 + \mu)(1 - 2\mu)}, \text{ Lamé's constant};$$

$$G = \frac{E}{2(1 + \mu)}, \text{ modulus of elasticity in shear};$$

$$E = \text{Young's modulus}; \text{ and}$$

$$\mu = \text{Poisson's ratio}.$$

Four particular solutions of the differential equations will be used in the following discussions. Two of these, designated H and V , apply to the pavement slab, and two, designated H_1 and V_1 apply to the subgrade. The solutions will be used in pairs, the H solution for the pavement with the H_1 solution for the subgrade, etc. In the H and H_1 solutions points on the line $r = 0$ move in a horizontal direction and in the V and V_1 solutions these points move in a vertical direction. In the following all subscripts of unity except that designating a Bessel

function of the first order refer to the subgrade. (J_0 and J_1 are Bessel functions.) The solutions are as follows:

$$\begin{aligned} u &= Z \sin \theta \cos pt \left[J_0(ar) - \frac{J_1(ar)}{ar} \right] \\ v &= Z \cos \theta \cos pt \frac{J_1(ar)}{ar} \\ w &= Z' \sin \theta \cos pt J_1(ar) \end{aligned} \quad (H)$$

$$\begin{aligned} u_1 &= Z_1 \sin \theta \cos pt \left[J_0(ar) - \frac{J_1(ar)}{ar} \right] \\ v_1 &= Z_1 \cos \theta \cos pt \frac{J_1(ar)}{ar} \\ w_1 &= -Z_1' \sin \theta \cos pt J_1(ar) \end{aligned} \quad (H_1)$$

$$\begin{aligned} u &= Z \cos pt J_1(ar) \\ v &= 0 \\ w &= -Z' \cos pt J_0(ar) \end{aligned} \quad (V)$$

$$\begin{aligned} u_1 &= Z_1 \cos pt J_1(ar) \\ v_1 &= 0 \\ w_1 &= Z_1' \cos pt J_0(ar) \end{aligned} \quad (V_1)$$

where $Z = M \cosh mz + M' \sinh mz + N \cosh nz + N' \sinh nz$;

$$\begin{aligned} Z' &= \frac{Mm}{a} \sinh mz + \frac{M'm}{a} \cosh mz + \frac{Na}{n} \sinh nz \\ &\quad + \frac{N'a}{n} \cosh nz; \end{aligned}$$

$$Z_1 = \operatorname{Re}[M_1 e^{-m_1 z} + N_1 e^{-n_1 z}];$$

$$Z_1' = \operatorname{Re}\left[\frac{M_1 m_1}{a} e^{-m_1 z} + \frac{N_1 a}{n_1} e^{-n_1 z}\right];$$

$$m = a\sqrt{1 - a^2}, \quad m_1 = a\sqrt{1 - a_1^2};$$

$$n = a\sqrt{1 - b^2}, \quad n_1 = a\sqrt{1 - b_1^2};$$

$$a = \frac{p}{a} \sqrt{\frac{\rho}{\lambda + 2G}}, \quad a_1 = \frac{p}{a} \sqrt{\frac{\rho_1}{\lambda_1 + 2G_1}};$$

$$b = \frac{p}{a} \sqrt{\frac{\rho}{G}}, \quad b_1 = \frac{p}{a} \sqrt{\frac{\rho_1}{G_1}};$$

$p = 2\pi$ times the frequency of vibration;

$\alpha = p/V = 2\pi/l$ where V is the radial velocity which the wave approaches asymptotically as it gets farther from its source and l is the corresponding wave length; and

M, N, M', N', M_1 , and N_1 are constants proportional to the amplitude of vibration.

$Re [\quad]$ signifies the real part of the expression in the bracket if either m_1 or n_1 is imaginary in a mathematical sense. If both m_1 and n_1 are real, then the Re in front of the brackets may be disregarded.

In each pair of solutions, for example, H and H_1 , the frequency $p/2\pi$, the velocity p/α , the six amplitude constants M, N, M', N', M_1 , and N_1 , and the physical properties are arbitrary. That is, the differential equations are satisfied for any arbitrary values of these parameters. However, the boundary requirements at the top of the pavement and at the common boundary between pavement and subgrade permit the elimination of the six amplitude constants. The result is an equation, called the frequency equation, giving the relation between frequency, velocity of wave propagation and the physical properties of pavement and subgrade. It is of interest that the frequency equation is the same whichever pair of solutions is used and depends only on the assumptions made in regard to boundary conditions. In the derivation which follows the assumption is made that the top of the pavement is free of force and that the boundary stresses and displacements of the pavement are equal to those of the subgrade at their common boundary.

DERIVATION OF THE FREQUENCY EQUATION

The plane $z = 0$ is taken in the middle of a slab of thickness $2c$ and the direction $\theta = 0$ is taken as due east as shown in Fig. 1.

The assumed boundary requirements result in the following relations:*

1. The top of the pavement is free of vertical force. $\sigma_z = 0$ at $z = -c$.

2. The top of the pavement is free of radial and tangential forces.

$$\tau_{rz} = \tau_{\theta z} = 0 \text{ at } z = -c.$$

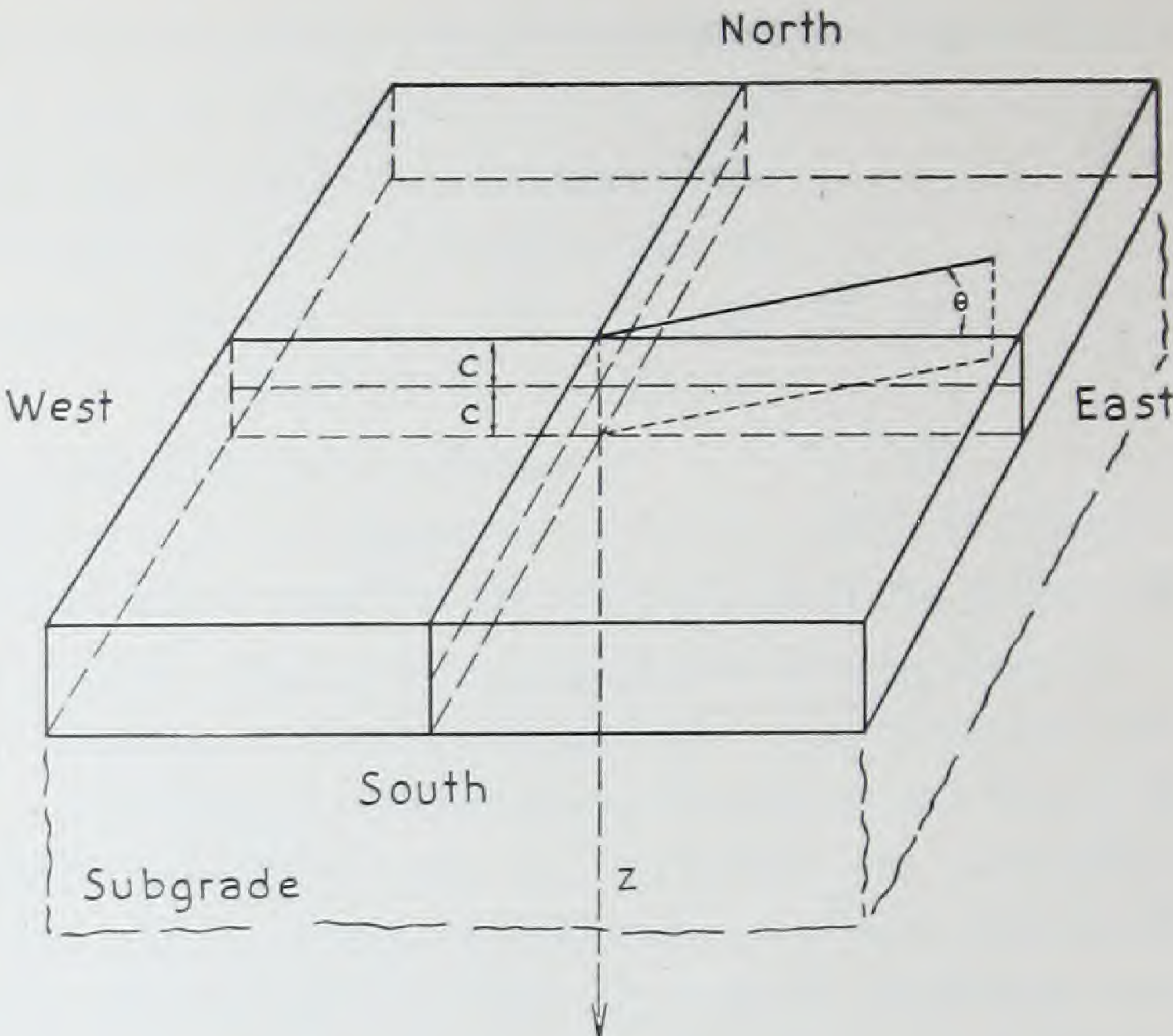
3. At the common boundary the vertical displacement of the pavement equals the vertical displacement of the subgrade. $w = w_1$ at $z = c$.

4. At the common boundary the radial and tangential displacements of the pavement equal the radial and tangential displacements, respectively, of the subgrade. $u = u_1$ and $v = v_1$ at $z = c$.

5. At the common boundary the vertical normal stresses are equal. $\sigma_z = \sigma_{1z}$ at $z = c$.

*Two relations are given in each of requirements 2, 4, and 6; but in each case the second relation is satisfied if the first is satisfied. Therefore the two relations give only one independent equation.

Fig. 1—Element of pavement slab and subgrade



6. At the common boundary the boundary shear stresses are equal.
 $\tau_{rz} = \tau_{1rz}$ and $\tau_{\theta z} = \tau_{1\theta z}$ at $z = c$.

When either pair of solutions for displacements is used and use is made of the relations previously given between $m, n, \lambda, G, a, b, m_1, n_1, \lambda_1, G_1, a_1$, and b_1 and of the following relations between stresses and displacements,

$$\begin{aligned}\sigma_z &= \lambda \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) + 2G \frac{\partial w}{\partial z} \\ \tau_{rz} &= G \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right) \\ \tau_{\theta z} &= G \left(\frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \right) ,\end{aligned}$$

the six boundary relations listed above result in the following six equations, respectively:

$$(2 - b^2)[M \cosh mc - M' \sinh mc] + 2 [N \cosh nc - N' \sinh nc] = 0 \dots (1)$$

$$\begin{aligned}\frac{2m}{a} [- M \sinh mc + M' \cosh mc] \\ + (2 - b^2) \frac{a}{n} [- N \sinh nc + N' \cosh nc] = 0 \dots \dots \dots (2)\end{aligned}$$

$$\begin{aligned}\frac{m}{a} [M \sinh mc + M' \cosh mc] + \frac{a}{n} [N \sinh nc + N' \cosh nc] \\ = - Re \left[\frac{M_1 m_1}{a} e^{-m_1 c} + \frac{N_1 a}{n_1} e^{-n_1 c} \right] \dots \dots \dots (3)\end{aligned}$$

$$M \cosh mc + M' \sinh mc + N \cosh nc + N' \sinh nc \\ = \operatorname{Re} [M_1 e^{-m_1 c} + N_1 e^{-n_1 c}] \dots\dots\dots (4)$$

$$(2 - b^2) [M \cosh mc + M' \sinh mc] + 2[N \cosh nc + N' \sinh nc] \\ = \frac{G_1}{G} \operatorname{Re} [(2 - b_1^2) M_1 e^{-m_1 c} + 2N_1 e^{-n_1 c}] \dots\dots\dots (5)$$

$$\frac{2m}{a} [M \sinh mc + M' \cosh mc] + (2 - b^2) \frac{a}{n} [N \sinh nc + N' \cosh nc] \\ = - \frac{G_1}{G} \operatorname{Re} \left[2 \frac{M_1 m_1}{a} e^{-m_1 c} + (2 - b_1^2) \frac{a}{n_1} N_1 e^{-n_1 c} \right] \dots\dots\dots (6)$$

The elimination of the six amplitude constants M , M' , N , N' , M_1 , and N_1 from these six equations gives the following frequency equation:

$$\frac{h_1 f_1 (1 - A_1 - A_1')}{1 - A_2 - A_2'} - \frac{1 - A_3 - A_3'}{1 - A_4 - A_4'} = 0 \dots\dots\dots (A)$$

where $A_1 = \phi \left[\frac{2 - b^2}{2} \coth mc - k \coth nc + \frac{2 - b_1^2}{2} \frac{b^2}{2} \frac{\sqrt{1 - a^2}}{f_1} \right];$

$$A_2 = \phi \left[\frac{2 - b_1^2}{2} \frac{2 - b^2}{2} \coth mc - \frac{2 - b_1^2}{2} k \coth nc \right. \\ \left. + h_1 \frac{b^2}{2} \sqrt{1 - a^2} \right];$$

$$A_3 = \phi \left[\frac{2 - b_1^2}{2} \frac{2 - b^2}{2} \coth mc - \frac{2 - b_1^2}{2} k \coth nc \right. \\ \left. + f_1 \frac{b^2}{2} \sqrt{1 - b^2} \coth mc \coth nc \right];$$

$$A_4 = \phi \left[\frac{2 - b^2}{2} \coth mc - k \coth nc \right. \\ \left. + \frac{2 - b_1^2}{2} \frac{b^2}{2} \frac{\sqrt{1 - b^2}}{h_1} \coth mc \coth nc \right];$$

$$\phi = \frac{G_1}{2G \left[\left(\frac{2 - b^2}{2} \right)^2 \coth mc - k \coth nc \right]};$$

$$k = \sqrt{(1 - a^2)(1 - b^2)};$$

$$h_1 = \sqrt{1 - b_1^2} \quad \text{if } b_1^2 < 1, \quad \text{i.e., if } n_1 \text{ is real}$$

$$h_1 = -\sqrt{b_1^2 - 1} \cot(ac\sqrt{b_1^2 - 1}) \quad \text{if } b_1^2 > 1, \text{ i.e., if } n_1 \text{ is imaginary}$$

$$f_1 = \sqrt{1 - a_1^2} \quad \text{if } a_1^2 < 1, \quad \text{i.e., if } m_1 \text{ is real;}$$

$$f_1 = \sqrt{a_1^2 - 1} \tan(ac\sqrt{a_1^2 - 1}) \quad \text{if } a_1^2 > 1, \text{ i.e., if } m_1 \text{ is imaginary.}$$

The expressions for A_1' , A_2' , A_3' , and A_4' used in the frequency equation are the same as for the expressions for A_1 , A_2 , A_3 , and A_4 , respectively, except that in every instance \coth is replaced by \tanh .

THE MEANING OF IMAGINARY AND COMPLEX VALUES FOR m , n , m_1 , AND n_1

The question of real and imaginary quantities entered into the solutions for the subgrade. The reason for this was that certain of the expressions became complex (contained both real and imaginary parts) under some conditions.

If the velocity of propagation of the waves horizontally is greater than the normal travel of disturbances within the subgrade, then either n_1 or both n_1 and m_1 become either imaginary or complex quantities in a mathematical sense. If internal friction is neglected as in the present study, then they become imaginary, but if internal friction is taken into account they become complex quantities. That is, if in the foregoing development p/a is greater than $\sqrt{G_1/\rho_1}$, then b_1 will be greater than unity and n_1 will be imaginary, and if p/a is also greater than $\sqrt{(\lambda_1 + 2G_1)/\rho_1}$, then m_1 will also be imaginary. When n_1 and m_1 are imaginary, the expressions $e^{-n_1 z}$ and $e^{-m_1 z}$ become trigonometric functions of the depth with both real and imaginary parts. For example, the real part of the expression for Z_1 becomes

$$Z_1 = M_1 \cos (az \sqrt{a_1^2 - 1}) + N_1 \cos (az \sqrt{b_1^2 - 1})$$

and for Z_1' becomes

$$Z_1' = M_1 \sqrt{a_1^2 - 1} \sin (az \sqrt{a_1^2 - 1}) - N_1 \frac{\sin (az \sqrt{b_1^2 - 1})}{\sqrt{b_1^2 - 1}}$$

The factor n may also be imaginary, but such a possibility does not introduce any ambiguity into the solutions since $\cosh nz$ and $(\sinh nz)/n$ are both real whether n is real or imaginary. The factor m will ordinarily not be imaginary because the shear modulus of the pavement is greater than the shear modulus of the subgrade.

If either m_1 or n_1 is imaginary, then as just indicated, a part of the expression for a displacement within the subgrade will be a trigonometric function of the depth z . Consequently, this part of the expression will indicate no diminution of maximum amplitude of vibration with increase of depth in the subgrade. Had internal friction been taken into account, the solutions would indicate a decrease in amplitude with increase of depth. For example, if the assumption is made that internal friction is adequately considered by replacing the usual Hooke's law by equations of the type*

*The term with f has been added to the usual expression for Hooke's law such as given by Timoshenko in *Theory of Elasticity* (New York: McGraw-Hill, 1934), pp. 7-10.

$$e_x + f \frac{\partial e_x}{\partial t} = \frac{1}{E} \left[\sigma_x - \mu \sigma_y - \mu \sigma_z \right]$$

$$\gamma_{xy} + f \frac{\partial \gamma_{xy}}{\partial t} = \frac{\tau_{xy}}{G}$$

then solutions corresponding to those given in V_1 above for the subgrade can be written

$$u_1 = Z_1 \sin \Theta \operatorname{Re} \left[e^{ipt} \left\{ H_0^{(1)}(ar) - \frac{H_1^{(1)}(ar)}{ar} \right\} \right]$$

with similar expressions for v_1 and w_1 , where the "real" Bessel functions of the first kind (J_0 and J_1) have been replaced by "complex" Bessel functions of the third kind ($H_0^{(1)}$ and $H_1^{(1)}$)*; the "real" function $\cos pt$ has been replaced by the "complex" function e^{ipt} ; and m_1 and n_1 have the values

$$m_1 = a \sqrt{1 - \frac{a_1^2}{1 + ifp}}, \quad n_1 = a \sqrt{1 - \frac{b_1^2}{1 + ifp}}$$

in place of the values given previously.

A separation of m_1 and n_1 into their real and imaginary parts shows that each has a positive real part for all values of a_1 and b_1 . Therefore, when internal friction of the subgrade is taken into account, the amplitude of vibration decreases with increase of depth.

In the above type of solution for vibration with internal friction the displacements and stresses are discontinuous at the line $r = 0$. It is on this line that the energy necessary to maintain sustained vibration is assumed to be supplied. To restrict the driving force to points on the pavement rather than at the line $r = 0$ would necessitate a still more complicated solution.

Since it is believed that internal friction has only a small effect on the velocity of propagation of the waves and since it was desired to keep the analysis relatively simple, friction was neglected in the derivation of the frequency equation.

NUMERICAL SOLUTION OF THE FREQUENCY EQUATION

The frequency equation derived above expresses the relation between the velocity† of wave propagation p/a , the frequency of vibration $p/2\pi$, and the physical properties of the pavement and subgrade. Unfortunately, this relation (Equation A) is rather involved, and numerical solution is not made readily. The method of solution found to be best in general was as follows:

*Bessel functions of first, second, and third kinds are treated in *Tables of Functions*, by Jahnke and Emde, 3d Ed. (New York: G. E. Stechert & Co., 1938), pp. 126-268.

†It must be remembered that the velocity of waves traveling radially depends on distance from the source. p/a is the velocity that they approach with increase of distance.

First, assume values for G_1/G , λ/G , λ_1/G_1 , ρ_1/ρ .

Second, select a value of b , the ratio of velocity of propagation to $\sqrt{G/\rho}$.

Third, determine a , a_1 , and b_1 and the functions of these quantities and of b that will be needed later such as $\sqrt{1 - b^2}$, $\sqrt{1 - b_1^2}$, $(2 - b^2)/2$, etc.

Fourth, select a value of ac/π , the ratio of frequency times pavement thickness to the velocity of wave propagation.

Fifth, perform all indicated substitutions into the frequency equation, Equation A.

Sixth, select a second value of ac/π and make the indicated substitutions in Equation A.

Seventh, continue the process of selecting values of ac/π until Equation A is satisfied to the desired accuracy.

Eighth, select other values of b and repeat the above procedure as many times as seems desirable.

Results of such a procedure are shown in Fig. 2 where the velocity of wave propagation is plotted against frequency of vibration times thickness of subgrade.

Fig. 2 shows that there are three possible wave velocities for each value of frequency times thickness. The highest of these velocities is almost the same as the velocity of longitudinal waves in the pavement, if it were free of the subgrade. The lowest of these velocities is almost the same as the velocity of transverse (flexural) waves in the pavement if it were free of the subgrade. The intermediate velocity is between that of "Rayleigh waves" in the subgrade alone and that of Rayleigh waves in the pavement. At high frequencies all three velocities approach the velocity of Rayleigh waves in the pavement.

The fact that the highest and lowest velocities are practically independent of the properties of the subgrade is of considerable importance. For practical purposes it may therefore be unnecessary to obtain numerical solutions to the foregoing frequency equation. Instead, the following frequency equations derived by Lamb and Timoshenko will be adequate:

For the higher velocity (longitudinal vibration)

$$\left(\frac{2 - b^2}{2}\right)^2 \coth mc = k \coth nc \dots\dots\dots (L)$$

and for the lower velocity (transverse vibration)

$$\left(\frac{2 - b^2}{2}\right)^2 \tanh mc = k \tanh nc \dots\dots\dots (T)$$

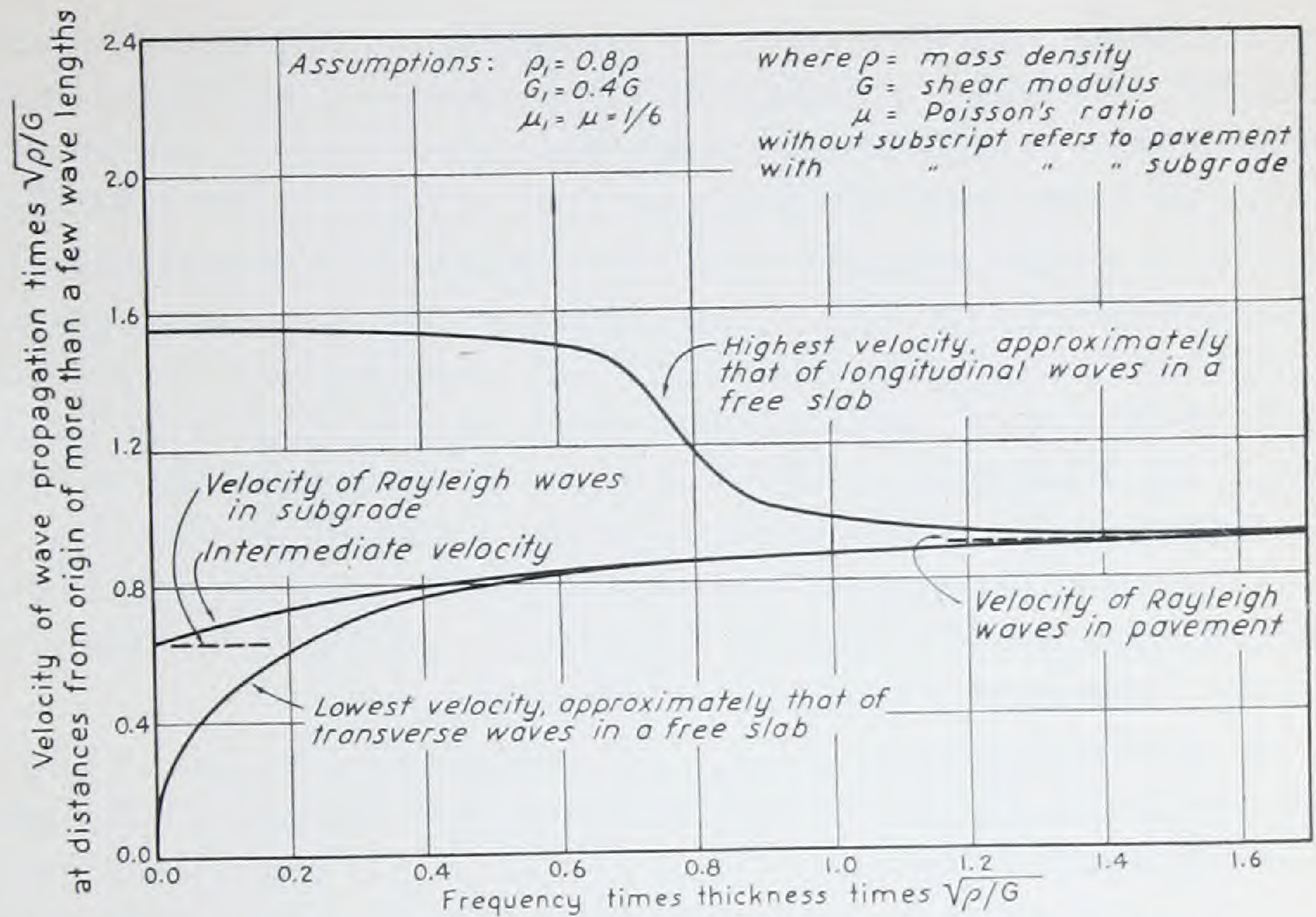


Fig. 2—Effect of frequency of vibration, thickness of pavement, and properties of subgrade on velocity of propagation of waves

These two frequency equations are readily obtained from Equation A by setting G_1 equal to zero. When this is done, either Equation L or T must be satisfied in order that a solution exist.

No correspondingly simple equation can be written for the approximate determination of the intermediate velocity. However, at low frequencies it is approximately the velocity of Rayleigh waves in the subgrade; at high frequencies it is approximately the velocity of Rayleigh waves in the pavement; and at intermediate frequencies it lies between these two limiting velocities. Therefore, a knowledge of these two limiting velocities may be all that is necessary for practical purposes. The relations are:

For Rayleigh waves in the subgrade

$$\left(\frac{2 - b_1^2}{2}\right)^2 = \sqrt{(1 - a_1^2)(1 - b_1^2)} \dots \dots \dots (R_1)$$

and for Rayleigh waves in the pavement

$$\left(\frac{2 - b^2}{2}\right)^2 = k \dots \dots \dots (R)$$

The foregoing analysis is based upon the assumption that the properties of the subgrade do not vary with depth. The properties of the subgrade, of course, do vary with depth; but, because of dampening, the properties at greater depths probably have no appreciable effect on the wave. It was shown by Love⁽¹⁰⁾ that the velocity of Rayleigh

waves decreased with frequency if the rigidity of the earth increased with depth. Therefore, one might expect in an actual test that the intermediate velocity would first decrease and later increase with increase of frequency rather than exhibit a continual increase as shown by Fig. 2.

Since for a given pavement on a given subgrade there are three possible wave velocities for each frequency, there may be some question as to the situation in a given test. It should also be kept in mind that for each of these wave velocities the motion may be that corresponding to either the H-solution or the V-solution or a combination of them. Final conclusions will probably have to await a study of test data, but it is believed that the type of vibration can be largely controlled by the manner of driving. This belief is based upon a study of the theoretical equations. This study indicates the following:

(1) If the driving force is directed horizontally, waves of the H-solution with the highest of the three velocities should predominate. This conclusion is based upon the fact that in this case points on the line $r = 0$ have only a horizontal motion and of relatively high amplitude. Relatively large motion of the point of application of the driving force and in the direction of the driving force is necessary for large energy input.

(2) If the horizontally directed force is not applied near the central plane of the slab, waves of the lowest of the three velocities will also be produced. These waves will also be of the H-solution. These lower-velocity waves should be produced to the exclusion of the waves of the higher velocity if the driver produces a couple rather than a resultant force. It is believed that with the driving force applied horizontally at the top of the pavement, waves of the highest rather than the lowest velocity will predominate.

(3) If the driving force is directed vertically, waves of the V-solution with the lowest of the three velocities should predominate. This type gives the largest relative vertical motion for points on the line $r = 0$.

(4) Waves of the intermediate velocity should be produced in all cases. Since their velocity is largely controlled by the properties of the subgrade, it is believed that their relative amplitude will be largely controlled by the dampening properties of the subgrade. Perhaps these waves will be most pronounced at low frequencies. It is believed that a rotary driving force such as would be produced by an unbalanced rotating mass would be most favorable for the production of these waves. That is, forces which produced motions for both H- and V-solutions out of phase with each other are believed to be best.

(5) Waves of the V-solution and of the highest of the three velocities could be produced by a pressure bulb inserted at the center of the pavement if the pressure fluctuated periodically.

It should be remembered that the velocities mentioned above are the limiting velocities which are approached asymptotically as the waves get farther from the source. Since velocity varies with distance from the source, the nodes are non-uniformly spaced, especially near the origin, and spaced differently for the different types of vibration. Moreover, in vibration according to the H-solution the character of the motion and the spacing of the nodes are different in the direction of the driving force, north-south directions in Fig. 1, from what they are in a direction at right angles to the driving force, east-west in Fig. 1. The location of the nodes will also depend on which displacements, horizontal or vertical, are being detected.

The table below giving $1/\pi$ times the roots of the Bessel functions involved should be helpful because the relative distances of the nodes from the origin should correspond to the tabular values given. For example, as is evident from the equations for displacement, the vertical displacement in the V-solution is zero at every distance r from the origin for which $J_0(ar)$ is zero.

1/π Times the Roots of Bessel Functions

Order of the Root	$J_0(x)$	$J_1(x)$	$\left[J_0(x) - \frac{J_1(x)}{x} \right]$
0	—	0	—
1	0.7655	1.220	0.5860
2	1.757	2.233	1.696
3	2.755	3.238	2.717
4	3.754	4.241	3.726
5	4.753	5.243	4.731
n^*	$n - 0.25$	$n + 0.25$	$n - 0.25$

*Where n is a large number.

As is evident from an examination of the equations for displacements, the amplitude of motion of the antinode decreases with distance from the origin. The rate of decrease depends on the angle θ for waves of the H-solution: for example, *in the direction* of the driving force the amplitude is approximately inversely proportional to the square root of the distance from the origin; *at right angles* to the driving force the amplitude is approximately inversely proportional to the three-halves power of the distance. In the V-solution the motion of the antinodes is approximately inversely proportional to the square root of the distance away from the origin *in any direction*. These facts should be helpful in interpreting results.

Based upon the foregoing theoretical analysis the following experimental procedure with equipment such as that described by Long et al. ⁽⁶⁾ is recommended.

1. Place driver so as to produce a vertical driving force to the top of the pavement slab.
2. With input to driver set at a given frequency determine location of nodes by moving "pick-up" on radial lines from the driver and noting phase shift and magnitude of response on oscilloscope. (One pair of plates of the cathode ray oscilloscope will be connected to the pick-up circuit and the other pair will be connected to the driver circuit.)
3. Repeat the test at various frequencies.
4. Either repeat the above with driver set so as to produce horizontal motion or use the Long et al. method of determining the velocity of longitudinal waves.
5. Plot velocity of waves versus frequency and compare with curves of Fig. 2.
6. Determine proper values of G/ρ and thickness of slab that will give best agreement with theoretical curves.
7. If data on the properties of the subgrade are desired and the waves of intermediate velocity have not been detected and their velocity determined, use a mechanical driver of relatively low frequency. This should produce the waves of intermediate velocity and of sufficient relative amplitude for detection.

Example

Suppose that at 1000 cycles per second the distances between two nodes (other than the first two nodes) with the driving force acting first vertically and then horizontally are 2.25 ft. and 7.00 ft., respectively. The corresponding velocities will be 4,500 ft. per sec. and 14,000 ft. per sec. ($V = 2l$ times frequency) and the ratio of the velocities will be $4500/14,000 = 0.321$. Assume the lower velocity to be that of transverse waves and the higher velocity to be that of longitudinal waves. An examination of Fig. 2 shows that the velocities of these waves have a ratio of 0.321 at a value of about 0.115 for the abscissas and that the corresponding values of the ordinates are approximately 1.55 and 0.50. This means that

$$\sqrt{G/\rho} = \frac{14,000}{1.55} = 9,030 \text{ ft/sec}$$

and

$$\text{thickness} = \frac{9,030 \times 0.115}{1,000} = 1.04 \text{ ft.}$$

$$\text{If density of pavement is } 150 \text{ lb. per ft}^3, \text{ then } \rho = 4.66 \frac{\text{lb. sec}^2}{\text{ft}}$$

$$\text{and } G = 4.66(9030)^2 = 380 \times 10^6 \text{ lb. per ft}^2 = 2.64 \times 10^6 \text{ psi}$$

$$\text{If } \mu = 1/6, \text{ then } E = 14/6 \times 2.64 \times 10^6 = 6.15 \times 10^6 \text{ psi}$$

A test at another frequency, say 2000 cycles per second, should result in approximately the same values for thickness and moduli for the pavement. If radically different values are obtained, then the waves have not been correctly identified. One velocity in at least one of the tests might have been the intermediate velocity.

If the intermediate velocities are not in accord with the middle curve of Fig. 2, then other curves based on other assumptions should be prepared for these velocities.

All of the above discussion has pertained to sustained vibration (stationary waves). Sustained vibration can be considered as the result of two equal continuous wave-trains (progressive waves) traveling in opposite directions. An adequate treatment of a single, impact-generated wave-train of finite length traveling away from a source is beyond the scope of this paper. Because of the finite length of the train and because many different frequencies are usually represented, not all parts travel at the same velocity and the wave-form changes as the wave proceeds. However, of practical importance is the fact that the higher velocity (upper curve, Fig. 2) is almost independent of frequency for low frequencies. Therefore, a wave-train of longitudinal waves of low frequency can be propagated with relatively little change of wave-form. It is probably because of this fact that the velocities of "longitudinal" waves in pavements have been determined successfully by measuring the time required for a short wave-train of longitudinal waves to travel between two points, the short wave-train being produced by an impact.⁽⁶⁾

The effects of variations in physical properties of the subgrade at different depths below the pavement, and the significance of the fact that the pavement does not extend indefinitely in a horizontal direction have not been considered. However, vibration of appreciable amplitude will probably not extend very far from the source either down into the subgrade or horizontally in the pavement owing to the effect of internal friction, especially at high frequencies. It is therefore believed that only the pavement within a few feet of the source and only the material immediately below the pavement will have an appreciable effect upon the wave velocity near the source when the frequency is relatively high.

If the character of the subgrade at considerable depth is desired, relatively low frequencies would be required. The above analysis may be inadequate for a study of the variation in properties of the subgrade with depth since no provision was made for such variation in the equation.

SUMMARY

Equations are derived for the combined vibration of pavement and subgrade. Numerical solution of the frequency equation shows that for a given slab on a given subgrade three wave velocities are possible for each frequency of vibration. The highest of these velocities is almost the same as that of longitudinal vibration of a free slab; the lowest, almost the same as that of transverse (flexural) vibration of a free slab; and the intermediate, somewhere between the velocity of Rayleigh waves in the uncovered subgrade and the velocity of Rayleigh waves in the pavement.

Suggestions are given for the production of any one of the three possible velocities to the virtual exclusion of the other two.

NOTATION (PARTIAL LIST)

r, θ, z = cylindrical coordinates

u, v, w = displacements in r, θ, z directions, respectively, of the point (r, θ, z) in the pavement

u_1, v_1, w_1 = displacements in r, θ, z directions, respectively, of the point (r, θ, z) in the subgrade

t = time

ρ, ρ_1 = mass densities of pavement and subgrade, respectively

$$\lambda = \frac{E}{(1 + \mu)(1 - 2\mu)}$$

$$\lambda_1 = \frac{E_1}{(1 + \mu_1)(1 - 2\mu_1)}$$

E, E_1 = Young's modulus for pavement and for subgrade, respectively

μ, μ_1 = Poisson's ratio for pavement and subgrade, respectively

G, G_1 = Shear modulus for pavement and subgrade, respectively

$\sigma_x, \sigma_y, \sigma_z$ = normal stresses

$\tau_{rz}, \tau_{\theta z}$, etc. = shear stresses

e_x = normal strain in x -direction

$\gamma_{rz}, \gamma_{\theta z}, \gamma_{xy}$ = shear strains

$p = 2\pi$ times frequency of sustained vibration

$\alpha = p/V = 2\pi/l$ where V is the radial velocity which the wave approaches asymptotically as it gets farther from its source and l is the corresponding wave length

c = half thickness of pavement

$$a = \frac{p}{\alpha} \sqrt{\frac{\rho}{\lambda + 2G}}, \text{ ratio of velocity of propagation to velocity of compressional waves in interior of pavement}$$

$$b = \frac{p}{\alpha} \sqrt{\frac{\rho}{G}}, \text{ ratio of velocity of propagation to velocity of shear waves in interior of pavement; ordinate of Fig. 2}$$

$$m = \alpha \sqrt{1 - a^2}$$

$$n = \alpha \sqrt{1 - b^2}$$

a_1, b_1, m_1, n_1 have the same definitions as a, b, m , and n , respectively, except that those with subscript unity refer to subgrade instead of pavement

$$k = \sqrt{(1 - a^2)(1 - b^2)}$$

$Re [\quad]$ = real part of expression in brackets

J_0, J_1 = Bessel functions of the first kind of order zero and unity, respectively

H_0^1, H_1^1 = Bessel functions of the third kind

REFERENCES

1. T. C. Powers, "Measuring Young's Modulus of Elasticity by Means of Sonic Vibration," Proc. Am. Soc. Testing Mats. V. 38, Part II, 460 (1938).
2. F. B. Hornibrook, "Application of Sonic Method to Freezing and Thawing Studies of Concrete," ASTM Bulletin, No. 101, 5 Dec. (1939).
3. W. T. Thomson, "Measuring Changes in Physical Properties of Concrete by the Dynamic Method," Proc. Am. Soc. Testing Mats. V. 40, 1113 (1940).

4. Leonard Obert and Wilbur I. Duvall, "Discussion of Dynamic Methods of Testing Concrete with Suggestions for Standardization," *Proc. Am. Soc. Testing Mats.* V. 41, 1053 (1941).
5. R. K. Bernhard, "Highway Investigation by Means of Induced Vibrations," *Engr. Exp. Station Series, Penn. State Coll.* V. 33, Bull. No. 49 (1939).
6. Bartlett G. Long, Henry J. Kurtz and Thomas A. Sandenaw, "An Instrument and a Technic for Field Determination of the Modulus of Elasticity and Flexural Strength of Concrete (Pavements)"; *ACI JOURNAL*, Jan. 1945; *Proceedings* V. 41, p. 217.
7. B. G. Long and H. J. Kurtz, "Effect of Curing Methods upon the Durability of Concrete As Measured by Changes in the Dynamic Modulus of Elasticity," *Proc. Am. Soc. Testing Mats.* V. 43, 1051 (1943).
8. Horace Lamb, "On Waves in an Elastic Plate," *Roy. Soc. Proc.* 1916-17, Ser. A, V. 93, p. 114.
9. S. Timoshenko, "Transverse Vibrations of Bars of Uniform Cross-Section," *Phil. Mag.* V. 43, 125 (1922) (Series 6).
10. A. E. H. Love, *Some Problems of Geodynamics* (Cambridge Univ. Press, 1911).
11. A. E. H. Love, *Mathematical Theory of Elasticity*, 4th ed. (Cambridge Univ. Press, 1927), p. 288.